Chapter 5

Introduction to Nonlinear Programming

A linear programming problem aims to maximize or minimize a linear function subject to the linear constraints. But in many cases, the objective function may not be linear or all/some of the constraints may not be linear or both the objective and the constraints may not be linear. Such an optimization problem is called a nonlinear programming problem (NLPP). In this chapter, we discuss about NLPP's. Nonlinear programming is the process of solving an optimization problem defined by a system of equalities and inequalities, collectively termed constraints, over a set of unknown real variables, along with an objective function to be maximized or minimized, where some of the constraints or the objective function are nonlinear. It is the sub-field of mathematical optimization that deals with problems that are not linear. A typical non-convex problem is that of optimizing transportation costs by selection from a set of transportation methods, one or more of which exhibit economies of scale, with various connectivity and capacity constraints. An example would be petroleum product transport given a selection or combination of pipeline, rail tanker, road tanker, river barge. Owing to economic batch size the cost functions may have discontinuities in addition to smooth changes. In experimental science, some simple data analysis (such as fitting a spectrum with a sum of peaks of known location and shape but unknown magnitude) can be done with linear methods, but in general these problems, also, are nonlinear. Typically, one has a theoretical model of the system under study with variable parameters in it and a model the experiment or experiments, which may also have unknown parameters. One tries to find a best fit numerically. In this case one often wants a measure of the precision of the result, as well as the best fit itself. Nonlinear programming (NP) involves minimizing or maximizing a nonlinear objective function subject to bound constraints, linear constraints, or nonlinear constraints, where the constraints can be inequalities or equalities. Example problems in engineering include analyzing design tradeoffs, selecting optimal designs, and computing optimal trajectories. Unconstrained nonlinear programming is the mathematical problem of finding a vector x that is a local minimum to the nonlinear scalar function f(x).

5.1 Basics of Non Linear Programming

5.1.1 Definition:

Let n, m, and p be positive integers. Let X be a subset of R^n, let f, g_i, and h_j be real-valued functions on X for each i in $\{1, \ldots, m\}$ and each j in $\{1, \ldots, p\}$, with at least one of f, g_i, and h_j being nonlinear. A nonlinear minimization problem is an optimization problem of the form

{\displaystyle {\begin{aligned}{\text{minimize }}&f(x)\\{\text{subject to }}&g_{i}(x)\leq 0{\text{ for each }}i\in \{1,\dotsc

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,m\}\\&h_{j}(x)=0{\text{ for each }}j\in \{1,\dotsc ,p\}\\&x\in X.\end{aligned}}}Minimize f(x)

Subject to $g_i(x) \leq$ for each i€{1,…,m}

$h_j(x)=0$ for each j€{1,…,p}

x€X

A nonlinear maximization problem is defined in a similar way.

5.1.2 Different types of Constraint sets:

- There are several possibilities for the nature of the constraint set, also known as the feasible set or feasible region.
- An *infeasible* problem is one for which no set of values for the choice variable satisfies all the constraints. That is, the constraints are mutually contradictory, and no solution exists; the feasible set is the empty set.
- A *feasible* problem is one for which there exists at least one set of values for the choice variables satisfying all the constraints.
- An *unbounded* problem is a feasible problem for which the objective function can be made to be better than any given finite value. Thus there is no optimal solution, because there is always a feasible solution that gives a better objective function value than does any give proposed solution.

5.1.3 Methods:

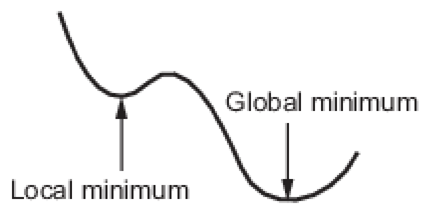
- If the objective function *f* is linear and the constrained space is a polytope, the problem is a linear programming problem, which may be solved using well-known linear programming techniques such as the simplex method.

- If the objective function is concave (maximization problem), or convex (minimization problem) and the constraint set is convex, then the program is called convex and general methods from convex optimization can be used in most cases.
- If the objective function is quadratic and the constraints are linear, programming techniques are used.
- If the objective function is a ratio of a concave and a convex function (in the maximization case) and the constraints are convex, then the problem can be transformed to a convex optimization problem using fractional programming techniques.

Several methods are available for solving non-convex problems. One approach is to use special formulations of linear programming problems. Another method involves the use of branch and bound techniques, where the program is divided into subclasses to be solved with convex (minimization problem) or linear approximations that form a lower bound on the overall cost within the subdivision. With subsequent divisions, at some point an actual solution will be obtained whose cost is equal to the best lower bound obtained for any of the approximate solutions. This solution is optimal, although possibly not unique. The algorithm may also be stopped early, with the assurance that the best possible solution is within a tolerance from the best point found; such points are called ε-optimal. Terminating to ε-optimal points is typically necessary to ensure finite termination. This is especially useful for large, difficult problems and problems with uncertain costs or values where the uncertainty can be estimated with appropriate reliability estimation. Under differentiability and constraint qualifications, the Karush–Kuhn–Tucker (KKT) conditions provide necessary conditions for a solution to be optimal. Under convexity, these conditions are also sufficient. If some of the functions are non-differentiable, sub differential versions of Karush–Kuhn–Tucker (KKT) conditions are available.

Local Optima and Global optima:

A local minimum of a function is a point where the function value is smaller than at nearby points, but possibly greater than at a distant point. A global minimum is a point where the function value is smaller than at all other feasible points.



If you need a global optimum, you must find an initial value for your solver in the basin of attraction of a global optimum. Suggestions for ways to set initial values to search for a global optimum:

- Use a regular grid of initial points.
- Use random points drawn from a uniform distribution if your problem has all its coordinates bounded. Use points drawn from normal, exponential, or other random distributions if some components are unbounded. The less you know about the location of the global optimum, the more spread-out your

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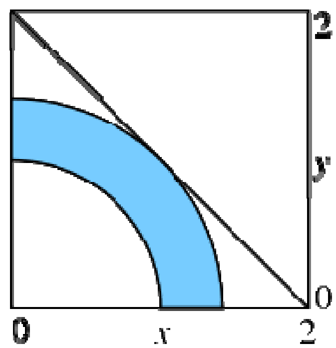
random distribution should be. For example, normal distributions rarely sample more than three standard deviations away from their means, but a Cauchy distribution (density $1/(\pi(1 + x^2)))$ makes hugely disparate samples.

- Use identical initial points with added random perturbations on each coordinate, bounded, normal, exponential, or other.

2-dimensional example:



The tangency of the line with the constrained space represents the solution. The line is the best achievable contour line (locus with a given value of the objective function).A simple problem can be defined by the constraints

$x_1 \geq 0$

$x_2 \geq 0$

$x_1^2 + x_2^2 \geq 1$

$x_1^2 + x_2^2 \leq 2$

with an objective function to be maximized

$f(\mathbf{x}) = x_1 + x_2$

where $\mathbf{x} = (x_1, x_2)$.

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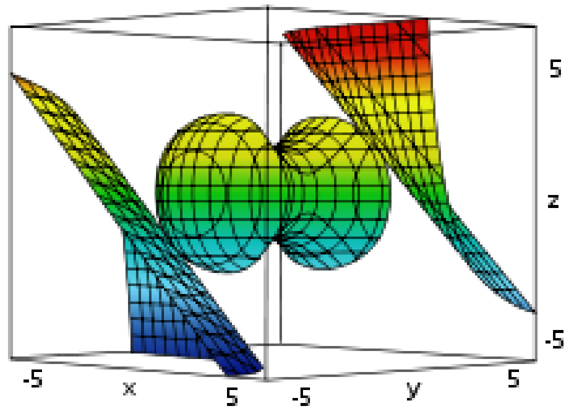
3-dimensional example:



The tangency of the top surface with the constrained space in the centre represents the solution. Another simple problem can be defined by the constraints

$$x_1^2 - x_2^2 + x_3^2 \leq 2$$

$$x_1^2 + x_2^2 + x_3^2 \leq 10$$

with an objective function to be maximized

$$f(\mathbf{x}) = x_1x_2 + x_2x_3$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

Given a transformation between input and output values, described by a mathematical function *f*, optimization deals with generating and selecting a best solution from some set of available alternatives, by systematically choosing input values from within an allowed set, computing the output of the function, and recording the best output values found during the process. Many real-world problems can be modeled in this way. For example, the inputs can be design parameters of a motor, the output can be the power consumption, or the inputs can be business choices and the output can be

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the obtained profit. The use of_optimization software_requires that the function *f* is defined in a suitable programming language and connected at compile or run time to the optimization software. The optimization software will deliver input values in *A*, the software module realizing *f* will deliver the computed value *f*(*x*) and, in some cases, additional information about the function like derivatives. Some of the common used software are AIMMS. Analytic Solver, ASTOS, CPLEX, General Algebraic Modeling System, LINDO, MATLAB etc.

5.2 Application of Nonlinear Programming

5.2.1 Application of derivatives to nonlinear programming for prescriptive analytics:

Besides the traditional probability theory and statistics, current machine learning techniques work in complete sync with linear algebra, graph theory, dynamic programming, multivariate calculus etc. As far as multivariate calculus is concerned, the different methods that lend support to machine learning algorithms are differential and integral calculus, partial derivatives, gradient and directional derivative, vector-valued function, Jacobian matrix and determinant, Hessian matrix, Laplacian and Lagrangian distributions etc. This blog will discuss the applications of second order derivatives and partial derivatives on optimization problems, as required for prescriptive analytics. Prescriptive analytics provides precise decisions on the course of action necessary for future success. One of the prominent applications of prescriptive analytics in marketing is the optimization problem of marketing budget allocation. The business problem is to figure out the optimum quantity of budget that needs to be allocated from the total

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advertising budget to each of the advertising media like TV, press, internet video etc. for maximizing the revenue. The budget optimization problem is solved either through Linear or Nonlinear Programming (NLP) which depends on whether:

1. The objective function is linear/ nonlinear
2. The feasible region is determined by linear/nonlinear constraints

Thus, one of the important assumptions for linear programming is the constant returns to scale for each of the advertising media. Partial derivative is the other prominent application of calculus on optimization problems. Partial derivatives is a function with several variables which are accomplished when a particular variable's derivative is computed keeping other variables constant. One of the most widely used applications of partial derivative is the least square criterion where the objective is to find out the best fitting line by minimizing the distance of the line from the data points. This is achieved by setting first order partial derivatives of the intercepts and the slopes equal to zero. The other major applications of partial derivatives are as follows:

- The second order partial derivative is used in an optimization problem to figure out whether a given critical point is a relative maximum, relative minimum, or a saddle point.
- The assignment of penalty for conversion of an optimization problem to an unconstrained optimization problem through sequential unconstrained minimization technique.

The approaches discussed show how calculus can be integrated with nonlinear programming while delivering an optimized solution. In the same manner, Lagrangean-based techniques can also be integrated with Mixed Integer Non-Linear Programming (MINLP) to provide with the marketing budget optimization solution. Currently, data scientists cannot bank on a single technique to provide analytics solution. The real challenge is to figure out how multiple techniques can be creatively combined to provide a solution as unique as the business problem.

5.2.2 Application of Non-linear programming models in Agriculture:

Experience with traditional programming models shows that a considerable improvement in performance is possible by adequately incorporating non-linear relationships. Particular emphasis will be given to the calibration and validation problems involved in this type of model. With the help of the Turkish agricultural sector model (TASM) it will be demonstrated that an empirical specification of a non-linear programming model for the agricultural sector is possible even with poor statistical data and that an operational model version can be handled.

A nonlinear programming model may be used for optimal planning and best use of nutrient ingredients. The solution of non-linear problems using penalization methods, which transforms the problem into an unconstrained non-linear optimization is oriented towards the study of Multi objective Optimization Problems. The production planning for examining the process of allocating land on agriculture activities and determining the optimal cropping patterns to achieve maximum in the income, and minima of irrigation water, use of fertilizers and number of workers non linear programming may be used.

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The discipline of agricultural economics has played a pioneering role in the application of mathematical models in economics. Originally, farm management specialists developed enterprise budgets to help farmers identify their best farming activities. These budgets, which specified the amounts of inputs required and outputs generated by allocating a unit of land to various crop or livestock production activities, facilitated the application of mathematical programming techniques as they were developed to offer normative advice to farmers. Eventually, programming models were expanded to offer advice at regional and national levels on issues as diverse as commercial agricultural policy, environmental policy, water and soil conservation policy, and public investment in infrastructure such as waterway and irrigation development. Agriculture was a natural application of stochastic and dynamic programming because of the stochastic nature of production and prices and the dynamic issues of capital investment and grain stock accumulation. On another front, agricultural economics played a key role in the development and application of econometrics for positive modeling purposes. With a wide variety of reasonably competitive product markets and abundant public data, agriculture presented a ready field of application for various statistical regression techniques as they were developed. Simultaneous equation models were easily expanded to address issues of international trade as they became important. In many cases, agricultural production characteristics such as heterogeneity, imperfect anticipation of product prices, and production risk motivated development and refinement of econometric techniques. With perceived competitive conditions, agriculture served as a

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ready laboratory for application of techniques that increasingly incorporated microeconomic theory in model specification, first with primal and then with dual approaches. In more recent times, as contracting has replaced open market transactions, agriculture has become a ready laboratory for application of game theory and the principal-agent models of mechanism design that now dominate microeconomic analysis.

Dynamic programming (DP) has shown itself to be an appropriate methodology for devising strategies of fisheries management. The model by Stanlfel (1988) has several goals in the same area: to elucidate the process of formulation and its requirements; to illustrate how practical constraints may be incorporated into the problems; to examine the realities of computation; to show that models of other than an equilibrium nature may be of use and that finite horizon models are more realistic; to emphasize the relevance of some models, not previously employed; and to correct some abuses and misconceptions. He also made some effort to reveal practical benefits to real-world decision-making in this problem area occasioned by the nature of DP. Certain significant advantages have been overlooked by previous researches.

Among the model improvements suggested are: backward formulations for more realistic decisions in both the cases of invertible and non-invertible transformations; the introduction of varying catch capacity and associated costs for changes and for underutilized capacity and the computational implications of multispecies models. For elementary deterministic models possible sources of error are identified and recommendations made for dealing with certain of these. A careful presentation was made of the effort

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involved in handling one-dimensional problems with random variables of known probability distributions. In the absence of distributions adaptive models are mentioned, along with their attending complications and some empirical evidence promoting optimism in that case.

To describe phytoplankton dynamics in the moving water there is a need of the characteristics of mixing and stirring since these processes determine the density distribution of phytoplankton. Since the characteristics of mixing in fluids depend on the control parameters (e.g. Reynolds number, etc.) many different situations can be studied, and since physical circumstances are so diverse in oceans and freshwaters, almost all possible situations must be studied. On the other hand, competitors form complex hierarchical networks in real ecosystems; networks may be built up by two elementary structures: transitive hierarchy (e.g. if B beats A and C beats B, then C beats A), and cyclic hierarchy (e.g. B beats A, C beats B, but A beats C). Thus, the dynamics of transitive and cyclical hierarchy either in open and closed chaotic flows (with and without coherent structures) or in turbulent flows to map the possible effects of interaction of population and hydrodynamics in competitive situations can be studied by non linear models.

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